

ON THE POWER-BOUNDED OPERATORS OF CLASSES C_0 AND C_1 .

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ABSTRACT. By a bounded backward sequence of the operator T we mean a bounded sequence $\{x_n\}$ satisfying $Tx_{n+1} = x_n$. In [8] we have characterized contractions with strongly stable nonunitary part in terms of bounded backward sequences.

The main purpose of this work is to extend that result to power-bounded operators.

Additionally, we show that a power-bounded operator is strongly stable (C_0 .) if and only if its adjoint does not have any nonzero bounded backward sequence. Similarly, a power-bounded operator is non-vanishing (C_1 .) if and only if its adjoint has a lot of bounded backward sequences.

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1. PRELIMINARIES

Let \mathcal{H} be a complex, separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear transformations acting on \mathcal{H} . By a *contraction* we mean $T \in \mathcal{B}(\mathcal{H})$ such that $\|Tx\| \leq \|x\|$ for each $x \in \mathcal{H}$. By a *power-bounded* operator we mean $T \in \mathcal{B}(\mathcal{H})$ such that $\|T^n\|$ is uniformly bounded for all $n = 1, 2, 3, \dots$

An operator T is said to be *completely nonunitary* (abbreviated *cnu*) if T restricted to every reducing subspace of \mathcal{H} is nonunitary. As usual, by T^* we mean the adjoint of T .

We define as usually:

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be of class C_0 if

$$\liminf_{n \rightarrow \infty} \|T^n x\| = 0$$

for each $x \in \mathcal{H}$.

Note that a power-bounded operator T is of class C_0 if and only if it is strongly stable ($T^n \rightarrow 0$, SOT).

Indeed, let T be C_0 .. If we fix $x \in \mathcal{H}$ then for each $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $\|T^k x\| < \epsilon$, so for all $m > k$ we have

$$\|T^m x\| = \|T^{m-k} T^k x\| \leq \|T^{m-k}\| \|T^k x\| \leq \epsilon \sup_{n \in \mathbb{N}} \|T^n\|.$$

¹Key words and phrases: power-bounded operators, C_0 operators, C_0 operators, strongly stable operators.

AMS(MOS) subject classification (2010): 47A05, 47A45, 47B37, 47G10.

In general, C_0 operators can be extremely different from strongly stable operators. The following example shows a bounded operator of class C_0 , which is not strongly stable at any (nonzero) point.

Example 1.2. Let $\{N_k\}_k$ be the sequence such that

$$\begin{cases} N_1 = 1 \\ N_{k+1} = 3N_k + 2N_k^2, \text{ for } k = 1, 2, \dots \end{cases}$$

Then let us define the operator S as the unilateral shift with weights w_1, w_2, \dots , i.e., $S : l^2 \ni (x_1, x_2, \dots) \mapsto (0, w_1x_1, w_2x_2, \dots) \in l^2$, where

$$\begin{cases} w_1 = 1 \\ w_i = \frac{1}{2}, \text{ for } i = N_k + 1, N_k + 2, \dots, 3N_k \\ w_i = 2^{\frac{1}{N_k}}, \text{ for } i = 3N_k + 1, 3N_k + 2, \dots, N_{k+1}. \end{cases}$$

For nonzero $x = (x_1, x_2, \dots) \in l^2$ there is i_0 such that $x_{i_0} \neq 0$. But by definition of S we have $S^{N_k}e_1 = e_{N_k+1}$ for all $k \in \mathbb{N}$, thus $\|S^{N_k-i_0}x\| \geq \|S^{N_k-i_0}x_{i_0}e_{i_0}\| = \frac{1}{w_1w_2\cdots w_{i_0}}|x_{i_0}|$.

So $S^n x \not\rightarrow 0$ for all nonzero $x \in l^2$.

Now we show that S is of class C_0 .

Fix $x = (x_1, x_2, \dots) \in l^2$ and $\epsilon > 0$. We can assume $\|x\| = 1$.

Since $\{N_k\}_k$ increases, then there is $N \in \{N_k | k = 1, 2, \dots\}$ such that

$\sum_{i=N+1}^{\infty} |x_i|^2 < \frac{\epsilon}{32}$ and $(\frac{1}{2^N})^2 < \frac{\epsilon}{2}$. By that we obtain:

$$\begin{aligned} \|S^{2N}x\|^2 &= \sum_{i=1}^N |x_i|^2 \|S^{2N}e_i\|^2 + \sum_{j=N+1}^{\infty} |x_j|^2 \|S^{2N}e_j\|^2 = \\ &= \sum_{i=1}^N |x_i|^2 \|S^N(w_i w_{i+1} \dots w_{i+N-1} e_{i+N})\|^2 + \sum_{j=N+1}^{\infty} |x_j|^2 |w_j w_{j+1} \dots w_{j+2N-1}|^2 \leq \\ &\leq \sum_{i=1}^N |x_i|^2 \|w_i w_{i+1} \dots w_N \cdot \left(\frac{1}{2}\right)^{i-1} S^N e_{i+N}\|^2 + \\ &+ \sum_{j=N+1}^{\infty} |x_j|^2 \left| \underbrace{2^{\frac{1}{N}} 2^{\frac{1}{N}} \dots 2^{\frac{1}{N}}}_{2N} \right|^2 \leq \sum_{i=1}^N |x_i|^2 \|S^N e_{i+N}\|^2 + \sum_{j=N+1}^{\infty} |x_j|^2 4^2 = \\ &= \sum_{i=1}^N |x_i|^2 \left\| \left(\frac{1}{2}\right)^N e_{i+2N} \right\|^2 + \frac{\epsilon}{32} 16 \leq \|x\| \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

In contrast to the above notion we have:

Definition 1.3. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be of class C_1 if

$$\liminf_{n \rightarrow \infty} \|T^n x\| > 0$$

for each nonzero $x \in \mathcal{H}$.

Operators of class C_1 are also called *non-vanishing*.

We also say that T is of class C_0 or C_1 if its adjoint is of class C_0 or C_1 , respectively.

Let us define $\mathcal{M}(T) := \{x \in \mathcal{H} \mid \exists \{x_n\}_{n \in \mathbb{N}} : x = x_0, Tx_{n+1} = x_n \text{ and } \{x_n\}_{n \in \mathbb{N}} \text{ is bounded}\}$. Naturally, such a sequence $\{x_n\}_{n \in \mathbb{N}}$ can be called as the *bounded backward sequence*.

2. INTRODUCTION

In the paper [8] we have presented the following theorem with some applications.

Theorem 2.1. *Let T be a contraction. The following conditions are equivalent:*

- *for any bounded backward sequence $\{x_n\}_{n \in \mathbb{N}}$ of T , the sequence of norms $\{\|x_n\|\}_{n \in \mathbb{N}}$ is constant,*
- *the nonunitary part of T is of class C_0 .*

We have been asked the natural question about a possible generalization to power-bounded operators. In this work we will try to answer this question.

The easy extension of the above theorem for power-bounded operators is not true. To see this, let us consider the following:

Example 2.2.

Let $T : l^2 \ni (x_1, x_2, x_3, \dots) \mapsto (0, x_1 + x_2, 0, x_3 + x_4, 0, \dots) \in l^2$.

It is clear that T is power-bounded, in fact $T = T^2$. Additionally, if $x = (a_1, a_2, a_3, \dots) \in \mathcal{M}(T) \subset T(\mathcal{H})$, then $a_{2k+1} = 0$ for all $k \in \mathbb{N}$. Hence $T^{-1}(\{x\}) = \{x\}$. Thus, even any (not necessary bounded) backward sequence of T must be constant.

On the other hand, T has trivial unitary part and is not C_0 , since $T^* = T^{*2}$.

3. CHARACTERIZATION OF C_1 AND C_0 POWER-BOUNDED OPERATORS

To introduce the next theorem, let us recall the construction of isometric asymptotes (see [7]).

Let us define a new semi-inner product on \mathcal{H} :

$$[x, y] := \text{glim} \{ \langle T^{*n}x, T^{*n}y \rangle \}_{n \in \mathbb{N}},$$

where glim denote a Banach limit.

Thus, the factor space $\mathcal{H}/\mathcal{H}_0$, where \mathcal{H}_0 stands for the linear manifold $\mathcal{H}_0 := \{x \in \mathcal{H} \mid [x, x] = 0\}$, endowed with the inner product $[x + \mathcal{H}_0, y + \mathcal{H}_0] = [x, y]$, is an inner product space. Let \mathcal{K} denote the resulting Hilbert space obtained by completion. Let X denote the natural embedding of Hilbert space \mathcal{H} into \mathcal{K} i.e. $X : \mathcal{H} \ni x \mapsto x + \mathcal{H}_0 \in \mathcal{K}$.

We can see that: $\|XT^*x\| = \|Xx\|$. So there is an isometry $V : \mathcal{K} \rightarrow \mathcal{K}$ such that $XT^* = VX$. The isometry V is called *isometric asymptote*.

Lemma 3.1. *For any power-bounded operator $T \in \mathcal{B}(\mathcal{H})$ the corresponding X from the construction above satisfies:*

$$X^*(\mathcal{K}) = \mathcal{M}(T).$$

Proof. By definition of V and X , we have $TX^* = X^*V^*$. Let $x_n := X^*V^n x$, then

$$Tx_{n+1} = TX^*V^{n+1}x = X^*V^*V^{n+1}x = x_n.$$

Moreover $\|x_n\| \leq \|X^*\| \|x\|$, for all $n \in \mathbb{N}$. Thus $X^*x = x_0 \in \mathcal{M}(T)$, where $x \in \mathcal{H}$. Hence $X^*(\mathcal{H}) \subset \mathcal{M}(T)$.

To prove the converse, let us fix $x \in \mathcal{M}(T)$. By definition of $\mathcal{M}(T)$, there exists $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$, a bounded backward sequence of x . Let $y \in \mathcal{H}$, then

$$(1) \quad |\langle x, y \rangle| = |\langle T^n x_n, y \rangle| = |\langle x_n, T^{*n} y \rangle| \leq \|x_n\| \|T^{*n} y\|.$$

So $|\langle x, y \rangle| \leq \sup_{n \in \mathbb{N}} \|x_n\| \liminf_{n \rightarrow \infty} \|T^{*n} y\| \leq \sup_{n \in \mathbb{N}} \|x_n\| \|Xy\|$. So by Theorem 1 in [9], we have $x \in X^*(\mathcal{K})$. \square

At the begining, we have observed that if $\liminf_{n \rightarrow \infty} \|T^{*n} x\| = 0$, for some $x \in \mathcal{H}$, then $\lim_{n \rightarrow \infty} \|T^{*n} x\| = 0$. So we have:

$$\{x \in \mathcal{H} | T^{*n} x \rightarrow 0\} = \mathcal{N}(X).$$

Now by Lemma 3.1 we obtain:

Corollary 3.2. *Let T be a power-bounded operator, then*

$$\mathcal{H} = \{x \in \mathcal{H} | T^{*n} x \rightarrow 0\} \oplus \overline{\mathcal{M}(T)}.$$

We also have:

Theorem 3.3. *A power-bounded operator T is C_1 if and only if $\overline{\mathcal{M}(T)} = \mathcal{H}$.*

It is trivial that $\mathcal{M}(T)$ is included in the set of origins of all backward sequences, that is, $T^\infty(\mathcal{H}) := \bigcap_{n \in \mathbb{N}} T^n(\mathcal{H})$. But in general, the converse inclusion does not hold, even for C_1 contractions.

To see this let us consider the following example.

Example 3.4. Let $\mathcal{H} = l^2$. Then $\mathbb{H} := l^2(\mathcal{H}) = \{\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H} | \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty\}$ is a separable Hilbert space, with the norm $\|\{x_n\}_{n \in \mathbb{N}}\| := \sqrt{\sum_{n \in \mathbb{N}} \|x_n\|^2}$.

For the element $\{x_n\}_{n \in \mathbb{N}} \in \mathbb{H}$ sometimes we write $\bigoplus_{n \in \mathbb{N}} x_n$.

Let S_w be the backward unilateral shift with weights $w = (w_1, w_2, \dots)$,

i.e., $S_w : \mathcal{H} \ni (x_1, x_2, \dots) \mapsto (w_1 x_2, w_2 x_3, \dots) \in \mathcal{H}$. If we put $w_i^n = (\frac{1}{n})^{\frac{1}{i-1} - \frac{1}{i}}$ for all $n \in \mathbb{N}$ and $i > 2$, and $w_1^n = w_2^n = 1$ for all $n \in \mathbb{N}$, then $T = \bigoplus_{n \in \mathbb{N}} S_{w^n}$ is a C_1 contraction.

Indeed, $T^m = \bigoplus_{n \in \mathbb{N}} S_{w^n}^m$, where $S_{w^n}^m(x_1, x_2, x_3, \dots) = (w_1^n w_2^n \dots w_m^n x_{m+1}, w_2^n w_3^n \dots w_{m+1}^n x_{m+2}, \dots)$ and $\lim_{m \rightarrow \infty} w_1^n w_2^n \dots w_m^n = \lim_{m \rightarrow \infty} (\frac{1}{n})^{\frac{1}{2} - \frac{1}{m}} = \frac{1}{n^2} > 0$.

Now, let us consider $x = \bigoplus_{n \in \mathbb{N}} (\frac{1}{n}, 0, 0, \dots) \in \mathbb{H}$.

For $m \in \mathbb{N}$ we have $x = T^m a_m$, where

$$a_m = \bigoplus_{n \in \mathbb{N}} \underbrace{(0, 0, \dots, 0)}_m, (\frac{1}{n})^{\frac{1}{2} + \frac{1}{m}}, 0, 0, \dots) \in \mathbb{H}. \text{ Thus } x \in T^\infty(\mathcal{H}).$$

Now let $\{b_m\}_{m \in \mathbb{N}} \subset \mathbb{H}$ be a backward sequence for x , then

$$x = T^m b_m \text{ and thus } b_m = \bigoplus_{n \in \mathbb{N}} (q_1, q_2, \dots, q_m, (\frac{1}{n})^{\frac{1}{2} + \frac{1}{m}}, 0, 0, \dots) \text{ for some}$$

complex q_1, q_2, \dots, q_m . So $\|a_m\| \leq \|b_m\|$, but

$$\|a_m\|^2 = \sum_{n \in \mathbb{N}} (\frac{1}{n})^{2(\frac{1}{2} + \frac{1}{m})} \rightarrow \infty \text{ (for } m \rightarrow \infty). \text{ Hence } x \notin \mathcal{M}(T).$$

One more consequence of Corollary 3.2 is the following:

Theorem 3.5. *A power-bounded operator T is C_0 if and only if $\mathcal{M}(T) = \{0\}$.*

Another proof of this theorem (in the case of operators considered on Banach spaces) can be found in [10].

Corollary 3.6. *If T is power-bounded and invertible, then*

$$\|T^{*n}x\| \rightarrow 0 \text{ for all } x \in \mathcal{H} \text{ if and only if } \|T^{-n}x\| \rightarrow \infty \text{ for all } x \in \mathcal{H}.$$

Proof. If T is power-bounded, then T^* is power-bounded too.

By Theorem 3.5 we obtain that T^* is strongly stable if and only if each nontrivial sequence such that $Tx_{n+1} = x_n$ is unbounded.

But we have $x_n = T^{-n}x_0$. Thus the second condition means that $\sup_{n \in \mathbb{N}} \|T^{-n}x\| = \infty$ for each nonzero $x \in \mathcal{H}$.

Now, if for some $x \in \mathcal{H}$ there is an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\sup_{k \in \mathbb{N}} \|T^{-n_k}x\| < N$, then for each $n \in \mathbb{N}$ we have

$$\|T^{-n}x\| = \|T^{n_k-n}T^{-n_k}x\| \leq \|T^{n_k-n}\| \|T^{-n_k}x\| \leq N \sup_{n \in \mathbb{N}} \|T^n\|,$$

since $n_k > n$ for some $k \in \mathbb{N}$. □

Example 3.7. Let V be the classical integral Volterra operator defined, on the space $L^2[0, 1]$, by

$$(Vf)(x) := \int_0^x f(t)dt, \text{ for } f \in L^2[0, 1].$$

It is easy to calculate that $(V^*f)(x) = \int_x^1 f(t)dt$.

Hence $V + V^* = P$, where P is the one-dimensional projection on

subspace of constant functions. It is well-known that $\|(I + V)^{-1}\| = 1$ (see Problem 150 in [4]). The Allan-Pedersen relation (see [1])

$$S^{-1}(I - V)S = (I + V)^{-1},$$

where $Sf(t) = e^t f(t)$ show us that $I - V$ is similar to a contraction. So it is power-bounded.

Furthermore, Proposition 3.3 from [5] yields to

$$(2) \quad \lim_{n \rightarrow \infty} \sqrt{n}(I - V)^n V f = 0, \text{ for all } f \in L^2[0, 1].$$

But as we mentioned, $I - V$ is power-bounded. Moreover, V has dense range. Therefore $I - V$ is C_0 . (To obtain this, instead of (2) we can use the Esterle-Katznelson-Tzafriri theorem (see [3], [6]), since $\sigma(I - V) = \{1\}$.)

Now, by Corollary 3.6 we obtain

$$\|(I + V - P)^{-n} f\| = \|((I - V)^*)^{-n} f\| \rightarrow \infty \text{ for all } f \in L^2[0, 1] \setminus \{0\}.$$

Additionally, from (1) in the proof of Lemma 3.1 and (2) we have

$$\frac{1}{\sqrt{n}} \|(I + V - P)^{-n} f\| \rightarrow \infty \text{ for all } f \in V(L^2[0, 1]) \setminus \{0\}.$$

Remark. To obtain the first part of this result we also can use Theorem 3.4 from [2] and observe that each local spectrum $\sigma_x(I + V - P)$ is equal $\{1\}$. (Because $\{1\} = \sigma(I - V) = \sigma((I + V - P)^*) = \sigma(I + V - P)$.)

Example 3.8. According to the above example we see that the contraction $(I + V)^{-1}$ is of class C_0 and as before $\sigma(I + V) = \{1\}$.

So using Theorem 3.4 from [2] we obtain that $\|(I + V)^n f\| \rightarrow \infty$ for all nonzero $f \in L^2[0, 1]$.

Now by Corollary 3.6 we have

$$(I - V + P)^{-n} f = (I + V)^{-n} f \rightarrow 0 \text{ for all } f \in L^2[0, 1].$$

So the contraction $(I + V)^{-1}$ is of class C_{00} .

4. MAIN RESULT

To give a generalization of Theorem 2.1, we will need the following lemma (due to Kérchy, see [7]):

Lemma 4.1. *If T is power-bounded, then T can be represented by the matrix*

$$(3) \quad \begin{bmatrix} T_{11} & T_{21} \\ 0 & T_{22} \end{bmatrix},$$

where T_{11}, T_{22} are power-bounded, T_{11} is of class C_0 and T_{22} is of class C_1 .

Proof. Let (3) be the matrix of T with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, where $\mathcal{N} := \{x \in \mathcal{H} \mid T^n x \rightarrow 0\}$. By definition \mathcal{N} is invariant for T . So $T|_{\mathcal{N}} = T_{11}$, thus T_{11} is of class C_0 (and power-bounded). Moreover, we have:

$$T_{22} = P_{\mathcal{K}} T \in \mathcal{B}(\mathcal{K}), \text{ where } \mathcal{K} := \mathcal{N}^\perp \neq \{0\}.$$

The subspace \mathcal{K} is invariant for T^* . So we obtain $T^*|_{\mathcal{K}} = T_{22}^*$, thus T_{22} is power-bounded.

Now, we will show that T_{22} is C_1 .

To see this, let us assume that $T_{22}^n f \rightarrow 0$ for some $f \in \mathcal{K}$.

For an arbitrary $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|T_{22}^{n_0} f\| < \frac{\epsilon}{2M}$, where $M := \sup_{n \in \mathbb{N}} \|T^n\|$.

Let us suppose for a while that $T_{22} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. By definition of T_{22} we have: $(T - T_{22})x = P_{\mathcal{N}}Tx \in \mathcal{N}$ for each $x \in \mathcal{K}$.

Hence for each $k \in \{1, 2, 3, \dots, n_0\}$ there exists $m_k \in \mathbb{N}$ such that $\|T^{m'+k-1}(T - T_{22})T_{22}^{n_0-k}f\| \leq \frac{\epsilon}{2n_0}$ for all $m' \geq m_k$.

Now, for $m := \max\{m_k \mid k = 1, 2, \dots, n_0\}$ we have:

$$\begin{aligned} \|T^{m+n_0}f\| &= \|T^m(T^{n_0} - T^{n_0-1}T_{22} + T^{n_0-1}T_{22} - T^{n_0-2}T_{22}^2 + \dots + \\ &+ TT_{22}^{n_0-1} - T_{22}^{n_0} + T_{22}^{n_0})f\| = \left\| \sum_{k=1}^{n_0} T^{m+k-1}(T - T_{22})T_{22}^{n_0-k}f + T^m T_{22}^{n_0}f \right\| \leq \\ &\leq \sum_{k=1}^{n_0} \|T^{m+k-1}(T - T_{22})T_{22}^{n_0-k}f\| + \|T^m T_{22}^{n_0}f\| \leq n_0 \frac{\epsilon}{2n_0} + M \frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Thus $T^n f \rightarrow 0$, contrary to $f \in \mathcal{K}$.

So T_{22} is of class C_1 . □

Now, we can give our generalization of Theorem 2.1:

Theorem 4.2. *Let T be a power-bounded operator. The following conditions are equivalent:*

- for any bounded backward sequence $\{x_n\}_{n \in \mathbb{N}}$ of T , the sequence of norms $\{\|x_n\|\}_{n \in \mathbb{N}}$ is constant;
- T can be decomposed as $T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & U \end{bmatrix}$, where U is a unitary and T_{11} is of class C_0 .

Proof. To the proof of the first implication, let $T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$ be the matrix form Lemma 4.1, where $T_{11} \in \mathcal{B}(\mathcal{H}_1)$ is C_0 and $T_{22} \in \mathcal{B}(\mathcal{H}_2)$ is C_1 . Now, \mathcal{H}_2 is invariant for T , thus $T_{22} = T|_{\mathcal{H}_2}$. Hence, each bounded backward sequence of T_{22} is bounded backward sequence of T . So, by our assumption T_{22} is an isometry on $\overline{\mathcal{M}(T_{22})}$. But by Theorem 3.3 we have $\overline{\mathcal{M}(T_{22})} = \mathcal{H}_2$. So T_{22} is an isometry.

Finally, it can be decomposed as $T_{22} = U \oplus S_+$, where U is unitary and S_+ is the unilateral shift. But T_{22} is C_1 . So we have $T_{22} = U$.

To prove the converse implication, let us assume that $\{x_n\}_{n \in \mathbb{N}}$ is the bounded backward sequence of T . Let $x_n = a_n + b_n$, where $a_n \in \mathcal{H}_1$ and $b_n \in \mathcal{H}_2$. We have:

$$T_{11}a_{n+1} + (T_{21}a_{n+1} + Ub_{n+1}) = Ta_{n+1} + Tb_{n+1} = Tx_{n+1} = x_n = a_n + b_n.$$

So $T_{11}a_{n+1} = a_n$ and $\|a_n\| \leq \|x_n\|$. It means that $\{a_n\}_{n \in \mathbb{N}}$ is a bounded backward sequence of T_{11} , but T_{11} is of class C_0 . So by Theorem 3.5 we obtain $a_n \equiv 0$. Thus $\|x_{n+1}\| = \|b_{n+1}\| = \|Ub_{n+1}\| = \|b_n\| = \|x_n\|$. \square

5. ACKNOWLEDGEMENTS

I am very grateful to professor J. Zemánek for his hospitality and discussions during my stay in the Institute of Mathematics of the Polish Academy of Sciences.

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